A • Notes – Formulas

The modeling process is a key step of conception. First, a crude modeling allows to validate (or not) the concept and identify the best combination of properties that maximize the performances. Then, a more precise modeling gives access to the right intervals for the forces, the displacements, the speeds, the heat fluxes, and the dimensions of the pieces. Eventually, the modeling provides precise values of the stresses, the strains and the probabilities of failure about the critical components, the power, the speed, the efficiency, and so on.

In this piece of work, some of the most current cases of geometry and simple loadings have been modeled. Most of the complex components may be modeled by approximating them as simple components through assumptions. There's no use to reinvent the beam, the column, or the pressure tank because their behavior under all the current loadings have already been studied. What is of importance here is to know that the result exists and where to find it.

This appendix contains the main useful results for the modeling process of regular problems. A lot of conception issues can be approached, with good accuracy, using the results that are given here. The detailed analysis of non critical components can also be done following the same pattern. Even if this method is not accurate enough in a particular case, the knowledge of the problem it bears stays valid.

This chapter is composed of 15 double pages which list, with commentaries, the results for:

- Constitutive equations,
- Bending behavior (beams), compression (columns), torsion (shafts),
- Contact stresses, cracks and other stress peaks,
- Pressure tanks,
- Beams/shells in vibration,
- Heat/material fluxes.
  These results come from many sources (refer to the bibliography).

The behavior of a component submitted to a load depends on the mechanism of strain. A beam in bending can deform elastically, plastically, by creeping, it can also fail under fragile or ductile failure. The equations which give the material response are known as the *constitutive equations.* Each mechanism is described by a different equation. These equations use one or more material properties:

- The Young modulus E and the Poisson ratio ν for the elastic strains,
- The yield point R_e for plastic strains,
- The creeping constants $\dot{\varepsilon}_0$, σ_0, and n,
- The fracture toughness K_{Ic} for the fragile failure.

The classical equations for the strain mechanism are listed here opposite. A multiaxial loading of principal stresses σ_1, σ_2, σ_3; where σ_1 is the biggest traction/compression stress is provided first for each equation. These equations are the basics in the determination of the mechanical response of materials.

Elastic strain

uniaxial	$\varepsilon_1 = \frac{\sigma_1}{E}$
general	$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{v}{E}(\sigma_2 + \sigma_3)$

Plastic strain

uniaxial	$\sigma_1 \geq R_e$
general	$(\sigma_1 - \sigma_3) = R_e$ (Tresca ➔ easy for hand calculus) $\sigma_e \geq R_e$ (Von Mises ➔ better for computational calculus) with $\sigma_e^2 = \frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$

Creep

uniaxial	$\dot{\varepsilon}_1 = \dot{\varepsilon}_0(\frac{\sigma}{\sigma_0})^n$
general	$\dot{\varepsilon}_1 = \dot{\varepsilon}_0 (\frac{\sigma^{n-1}}{\sigma_0^n})(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3))$

Failure

uniaxial	$\sigma_1 = \frac{CK_{Ic}}{\sqrt{\pi a}}$ with $\begin{cases} C = 1 \text{ in traction} \\ C = 15 \text{ in compression} \end{cases}$
general	$\sigma_1 = \frac{CK_{Ic}}{\sqrt{\pi a}}(\sigma_1 > \sigma_2 > \sigma_3)$

A uniform cross section beam loaded in simple traction by a force F, holds a stress $\sigma = F/A$, with A as the sectional area. The output is calculated using the constitutive equations here above. In this case, the important parameter of the section is its area. For other loading modes, one obtains higher moments of inertia. The moments for the usual sections are provided here below, namely:

- The moment of inertia measures the resistance of the section subjected to bending with respect to an horizontal axis (represented by the dotted lines). It is equal to:

$$I = \int_{section} y^2 b(y) dy$$

where y is measured vertically and $b(y)$ is the width of the section in y.

- The K moment measures the resistance of the section subjected to torsion. For circular sections, it is equal to the polar moment J defined by:

$$J = \int_{section} 2\pi r^3 dr$$

where r is measured from the center of the circular section and along the radius. For non-circular sections, K is lower than J.

- The section modulus $Z = I/y_m$ (with y_m as the distance between the neutral fiber in bending and the external surface of the beam) measures the surface stress generated by the bending moment M given by:

$$\sigma = \frac{My_m}{I} = \frac{M}{Z}$$

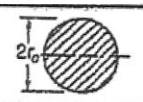
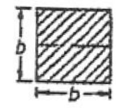
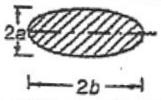
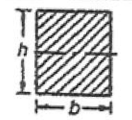
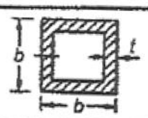
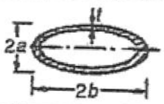
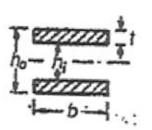
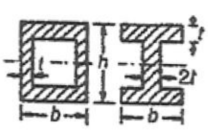
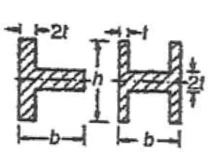
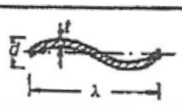
- Then, the H moment, defined by:

$$H = \int_{section} yb(y) dy$$

measures the resistance of the beam to total plasticity in bending. The moment of total plasticity for a beam in bending is:

$$M_p = HR_e$$

The thin or slender geometries can buckle before plastically deforming or failure. It is the buckling effect outbreak that fixes the thinness of the walls of tubes or of trellis.

Geometrv	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}r^3$	$\frac{\pi}{2}r^3$
	b^2	$\frac{b^4}{12}$	$0.14b^4$	$\frac{b^3}{6}$	$0.21b^3$
	πab	$\frac{\pi}{4}a^3b$	$\frac{\pi a^3b^3}{(a^2+b^2)}$	$\frac{\pi}{4}a^2b$	$\frac{\pi a^2b}{2}$ $(a<b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3h}{3}\left(1-0.58\frac{b}{h}\right)$ $(h>b)$	$\frac{bh^2}{6}$	$\frac{b^2h^2}{3h+1.8b}$ $(h>b)$
	$\frac{\sqrt{3}}{4}a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3 t$	$\frac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3 t$	$\frac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2 t$	$\frac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2\pi r^2 t$
	$4bt$	$\frac{2}{3}b^3t$	$b^3t\left(1-\frac{t}{b}\right)^4$	$\frac{4}{3}b^2t$	$2b^2t\left(1-\frac{t}{b}\right)^2$
	$\pi(a+b)t$	$\frac{\pi}{4}a^3t\left(1+\frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2}t}{(a^2+b^2)}$	$\frac{\pi a^2 t}{4}\left(1+\frac{3b}{a}\right)$	$2\pi t(a^3b)^{1/2}$ $(b>a)$
	$b(h_o-h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3-h_i^3)$ $\approx \frac{1}{2}bth_o^2$	—	$\frac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	—
	$2t(h+b)$	$\frac{1}{6}h^3t\left(1+\frac{3b}{h}\right)$	I: $\approx \frac{2tb^2h^2}{h+b}$; □: $\frac{2}{3}bt^3\left(1+\frac{4h}{b}\right)$	$\frac{h^2t}{3}\left(1+\frac{3b}{h}\right)$	I: $2tbh$; □: $\frac{2}{3}bt^2\left(1+\frac{4h}{b}\right)$
	$2t(h+b)$	$\frac{t}{6}(h^3+4bt^2)$	H: $\frac{t^3}{3}(8b+h)$; ⊢: $\frac{2}{3}ht^3\left(1+\frac{4b}{h}\right)$	$\frac{t}{3h}(h^3+4bt^2)$	H: $\frac{t^2}{3}(8b+h)$; ⊢: $\frac{2}{3}ht^2\left(1+\frac{4b}{h}\right)$
	$t\lambda\left(1+\frac{\pi^2d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	—	$\frac{t\lambda d}{4}$	—

When a beam is loaded by a force F or by a moment M, its axis initially straight deforms with a given curvature. If the beam has uniform properties and a uniform section and, if its length is high compared to its width and everywhere stressed elastically, the strain δ and the rotation angle θ can both be calculated by using the theory of elastic bending (see A.16). The basis differential equation describes the curvature of the beam in one point x of its length:

$$EI = \frac{dy^2}{d^2x} = M$$

with y as the lateral strain and M as the bending moment at point x. E is the Young modulus and I is the moment of inertia of the section (see A.2). When M is constant, the equation becomes:

$$\frac{M}{I} = E\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

with R_0 the curvature radius before the application of the moment and R the one that results of that operation. By integrating the equations over the beam, one finds the strain δ and the rotation angle θ. The expressions of these values are given below for usual loading modes.

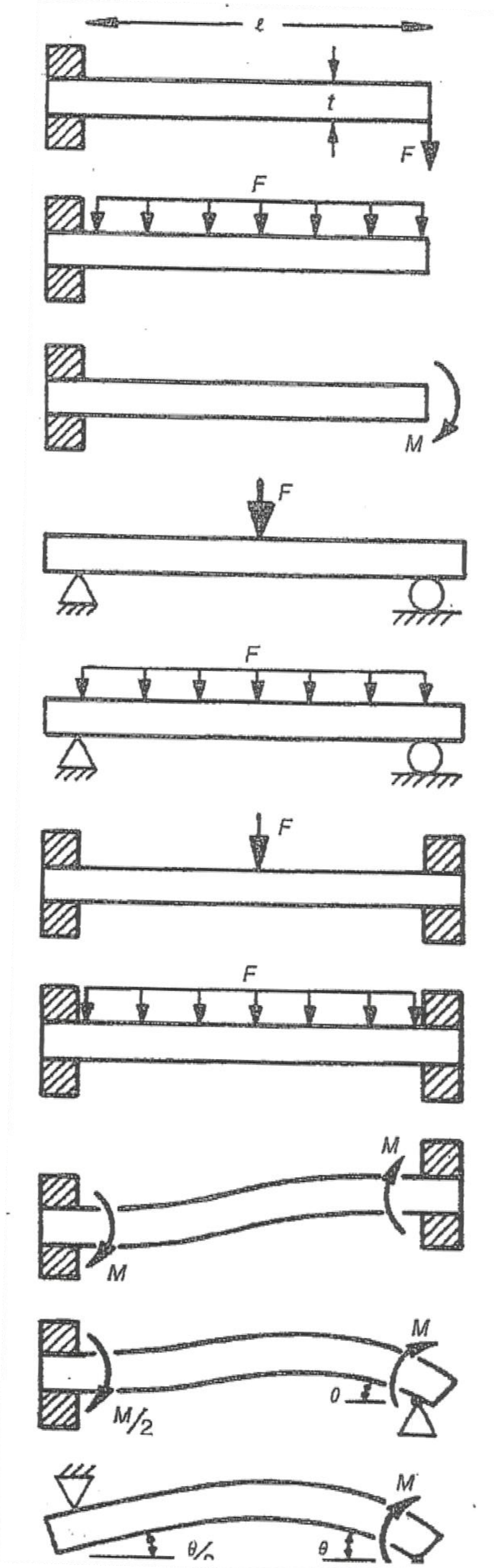
The stiffness of the beam is defined as:

$$S = \frac{F}{\delta} = \frac{C_1 EI}{I^3}$$

It depends on the Young modulus E of the material of the beam, of its length l and of its moment of inertia I of its section. The angle at the extremity θ is given by:

$$\theta = \frac{Fl^2}{C_2 EI}$$

The values of C_1 and C_2 are given below.



C_1	C_2
3	2
8	6
2	1
48	16
$\frac{384}{5}$	24
192	–
384	–
6	–
–	4
–	3

$$\delta = \frac{Fl^3}{C_1 EI} = \frac{Ml^2}{C_1 EI}$$

$$\theta = \frac{Fl^2}{C_2 EI} = \frac{Ml}{C_2 EI}$$

E = Young modulus

δ = strain

F = force (N)

M = moment (Nm)

ℓ = length (m)

b = width (m)

t = thickness (m)

θ = extremity angle

I = see table 2 (m^4)

y = distance of the F.N. (m)

R = curvature radius (m)

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

The longitudinal stress σ at a distant point of y of the neutral fiber of an uniform beam, loaded elastically in bending by a torque M, is:

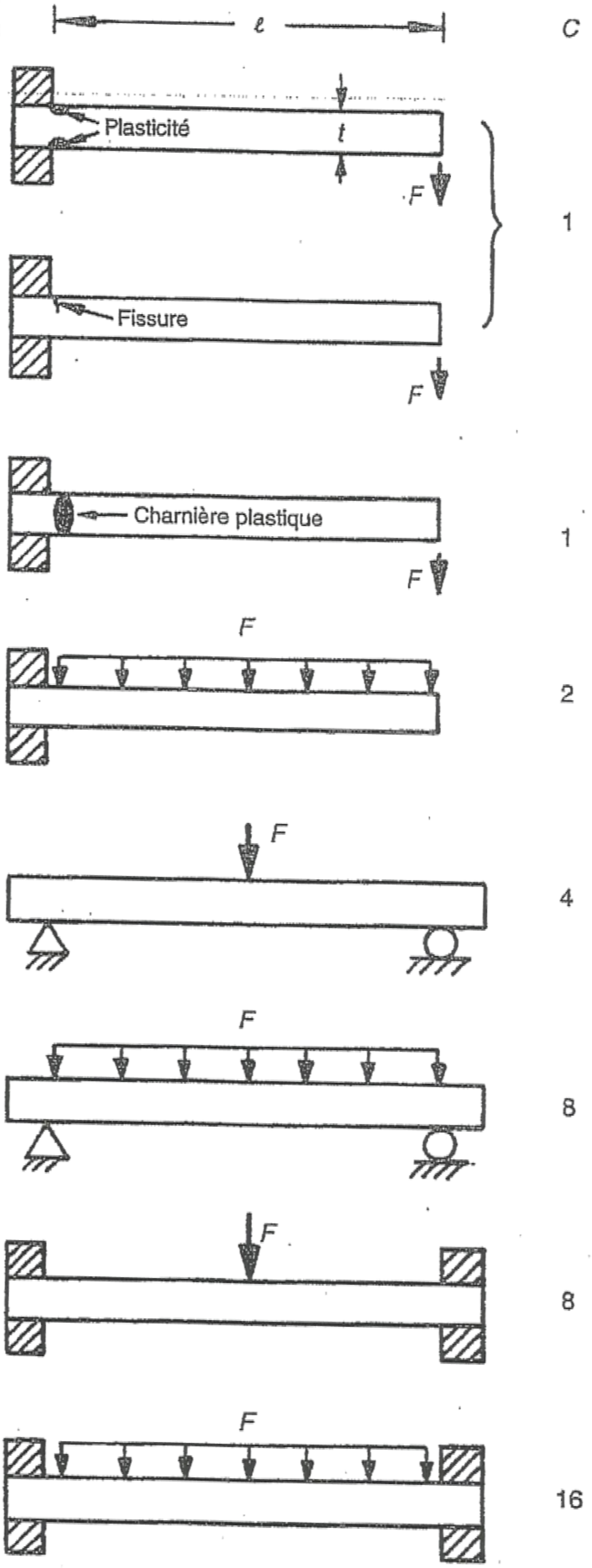
$$\frac{\sigma}{y} = \frac{M}{I} = E\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

with I as the moment of inertia of the section (see A.2); E as the Young modulus; R_0 as the initial curvature radius and R as the deformed curvature radius. The traction stress on the neutral fiber of this beam is:

$$\sigma = \frac{M y_m}{I} = \frac{M}{Z}$$

with y_m as the distance between the neutral fiber and the external surface of the beam. If the stress reaches the yield point R_e of the material of the beam, small zones of plasticity arise on the surface (see scheme at the top, here opposite). The beam is not elastic anymore and, in this meaning, is defaulting. If, on the contrary, the maximal longitudinal stress reaches the fragile failure stress (the failure modulus) of the material, a crack appears on the surface and propagates toward the inside (see second scheme); in this case, the beam is also defaulting. A third important criterion of defaulting should be considered: if the plastic zones penetrate toward the inside of the section of the beam and merge, they form a plastic hinge (see third scheme).

The torques and the loads, for each of these types of failure, and for each geometry and loading, are given here opposite. The equations with the mention “start” refer to the first two modes of failure; the ones with the mention “total plasticity” refer to the third type. In the first two modes, one can see the influence of Z, whereas the influence of H can be noticed in the third one (see section A.2).



$$M_f = \left(\frac{I}{y_m}\right)\sigma^* \; (Start)$$

$$M_f = H\sigma_y \; (Total\ plasticity)$$

$$F_f = C\left(\frac{I}{y_m}\right)\frac{\sigma^*}{l} \; (Start)$$

$$F_f = \frac{CH\sigma_y}{l} \; (Total\ plasticity)$$

M_f = moment (m)

F_f = force (N)

l = length (m)

t = thickness (m)

b = width (m)

I = see Table 2 (m^4)

$\frac{I}{y_m}$ = see Table 2 (m^3)

H = see Table 2 (m^3)

σ_y = yield stress (N/m^2)

σ_f = breaking stress (N/m^2)

$$\sigma^* = \begin{cases} \sigma_y & \text{(plastic material)} \\ \sigma_f & \text{(brittle material)} \end{cases}$$

If it is slender enough, a column loaded in compression will fail by an elastic buckling for a critical load F_{crit}. This load depends on the constraints on the extremities; four extreme cases are illustrated here opposite:

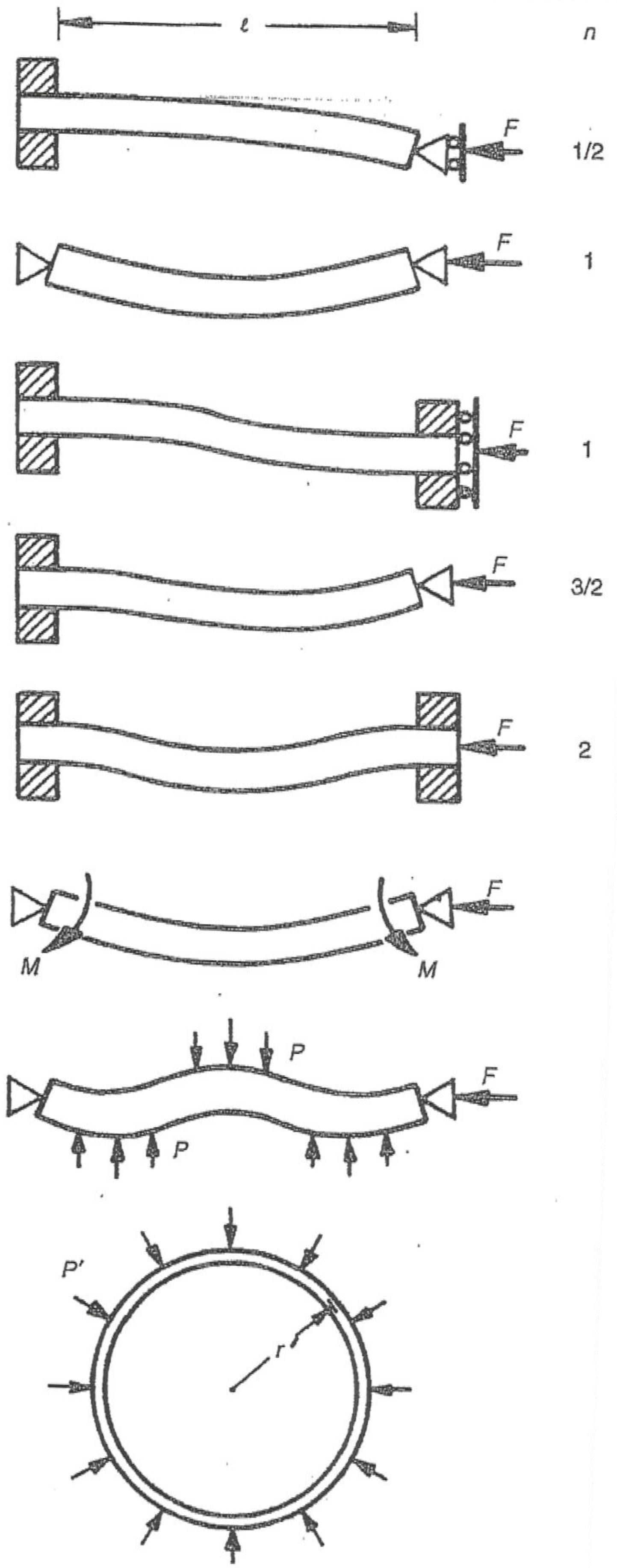
- An extremity can be constrained in position and in direction,
- It can be freed in rotation but not in translation, and conversely,
- It can be totally freed.

These constraints, taken two by two, lead to the five situations schematized here opposite, each of these characterized by a value of the constant n, which equals to the number of half wavelengths of the buckled form.

The addition of a torque M reduces the buckling load by the quantity expressed in the second box. A negative value of F_{crit} means that a traction force is necessary to avoid buckling.

An elastic plate applies a lateral recall pressure p proportional to its strain ($p = ky$ with k as the rigidity by depth unit and y as the local lateral strain). Its effect is to enhance the critical load of the quantity indicated in the third box.

An elastic thin walls tube will buckle elastically under the external pressure p' indicated in the last box. In this case, I is the moment of inertia of the longitudinal section of the tube.



$$F_{crit} = \frac{n^2\pi^2 E}{l^2}$$

Or

$$\frac{F_{crit}}{A} = \frac{n^2\pi^2 E}{(\frac{l}{r})^2}$$

M = moment (m)

F = force (N)

l = length (m)

A = sectional area (m^2)

I = see Table 2 (m^4)

r = gir. radius. $\left(\frac{I}{A}\right)^{1/2}$ (m)

k = rigidity of the plate (N/m^2)

n = half wavelength in buckling

p' = pressure (N/m^2)

$$F_{crit} = \frac{\pi^2 EI}{l^2} - \frac{M^2}{4EI}$$

$$F_{crit} = \frac{n^2\pi^2 EI}{l^2} + \frac{kl^2}{n^2}$$

$$p'_{crit} = \frac{3EI}{(r')^3}$$

A torque T applied to the extremities of an isotropic uniform section bar and acting in a perpendicular plane to the axis of the bar, produces a torsion angle θ. The torsion is linked to the value of the torque by the first equation below, in which G is the shear modulus. For the bars and the circular tubes, the K factor equals the polar inertia J (see A.2). Concerning other cross sections, $K < J$.

If the strain stops to be elastic, the shaft fails. It happens if the maximal surface stress exceeds either the yield stress R_e of the material or the failure stress. For the circular sections, the shear stress at all points distant of r from the rotation axis is:

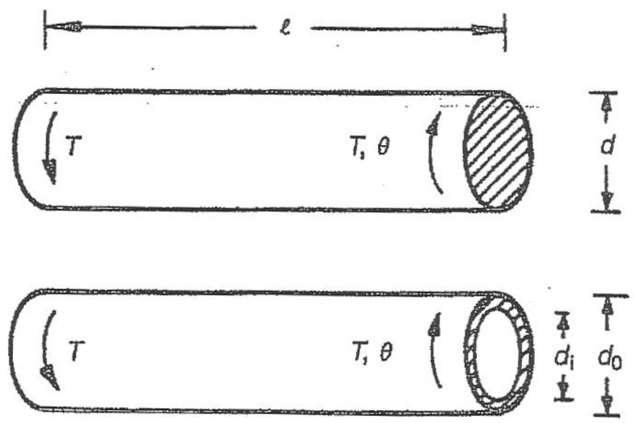
$$\tau = \frac{Tr}{K} = \frac{G\theta r}{l}$$

The shear stress and the traction stress are maximal at the surface and their values are:

$$\tau_{max} = \sigma_{max} = \frac{Td_0}{2K} = \frac{G\theta d_0}{2l}$$

If τ_{max} exceeds $R_e/2$ (Tresca criterion) or if σ_{max} exceeds the failure modulus, the bar fails, as shown in the figure. The maximal surface stress for filled sections which are ellipsoids, squares, rectangles, or triangles is at the surface point which is the nearest of the barycentre of the section. One can approximate it by evaluating the biggest circle inscribed in the section and by calculating the surface stress for a circular bar having the diameter of the circle. The geometries of more complex cross sections require specific thoughts and if they are thin, they can undergo buckling.

The helical springs are particular cases of torsion strain. The lengthening of each such springs of n helixes with a R radius under a force F and with the critical force F_{crit}, is shown here opposite.

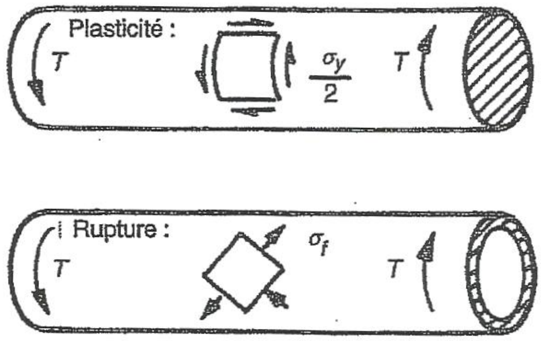


ETC

Elastic strain

$$\theta = \frac{lT}{KG}$$

Failure



$$T_f = \frac{K\sigma_y}{d_0} \text{ (start of plasticity)}$$

$$T_f = \frac{2K\sigma_f}{d_0} \text{ (fragile failure)}$$

T = Torque (N/m)

θ = Torsion angle

G = shear modulus (N/m^2)

l = length (m)

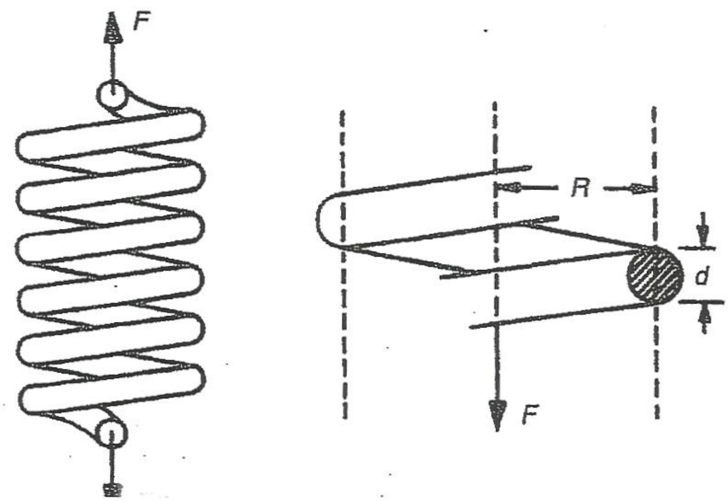
d = diameter (m)

K = see Table 1 (m^4)

σ_y = yield stress (N/m^2)

σ_f = failure modulus (N/m^2)

Spring



$$u = \frac{64FR^3n}{Gd^4}$$

$$F_f = \frac{\pi}{32}\frac{d^3\sigma_y}{R}$$

F = force (N)

u = displacement (m)

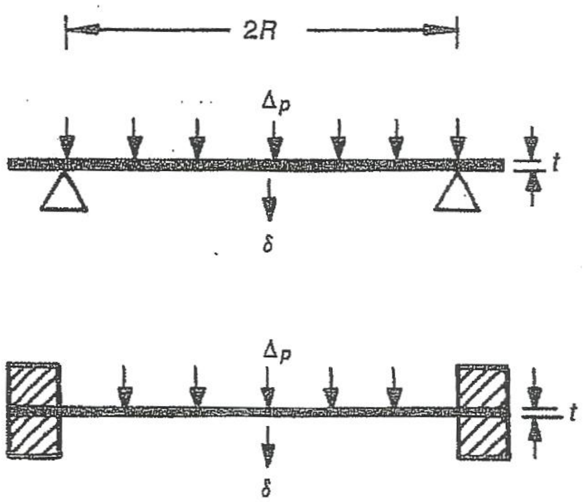
R = radius (m)

n = number of helixes

A thin disc deforms when one applies on it a pressure difference Δp between its two lateral surfaces. The strain induces stresses in the disc. The first box here below shows the expressions of the displacement and of the maximal stress (important to prevent failure) when the edges of the disc are simply supported. The second box gives the same expressions for a clamped disc.

The results for a thin horizontal disc deforming under its weight are found by replacing Δp by the surface masse $\rho g t$ of the disc (g is the intensity of the gravity ~ 9.81). The case of thick discs is more complicated (see bibliography).

The rotating discs, rings, and cylinder store the kinematic energy. The centrifugal forces induce stresses in the discs. The two boxes at the bottom show the expressions of the kinematic energy and of the maximal stress in the discs and the rotating rings at an angular speed ω (rad/s). Rotation speed and maximal energy are limited by the breaking resistance of the disc. One finds them by equaling the maximal stress in the disc with the resistance of the material.



Simply supported

$$\delta = \frac{3}{4}(1-\nu^2)\frac{\Delta p R^4}{E t^3}$$

$$\sigma_{max} = \frac{3}{8}(3+\nu)\frac{\Delta p R^2}{t^2}$$

Tighten up

$$\delta = \frac{3}{16}(1-\upsilon^2)\frac{\Delta p R^4}{E t^3}$$

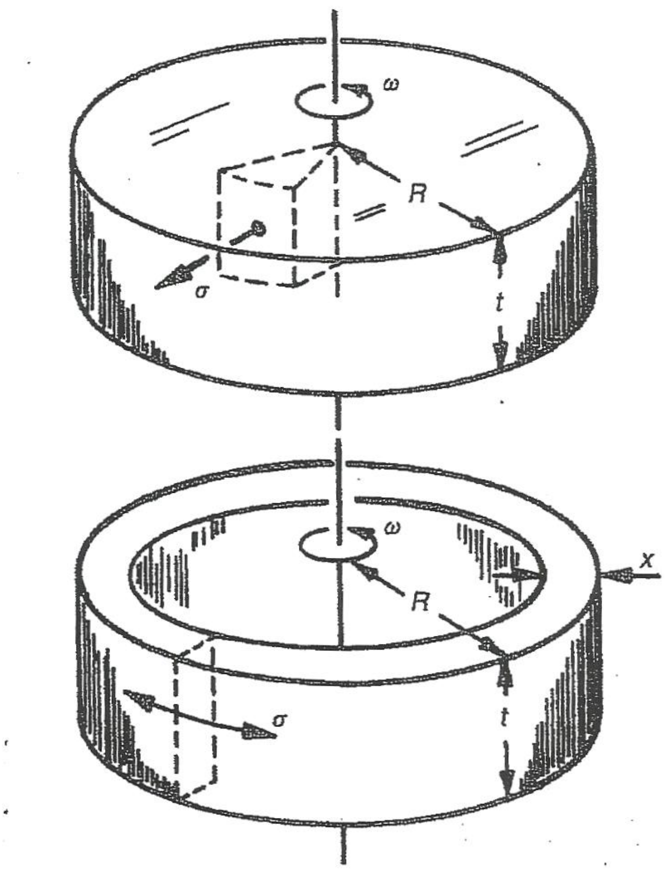
$$\sigma_{max} = \frac{3}{8}(1+\nu)\frac{\Delta p R^2}{t^2}$$

δ = displacement (m)

E = Young modulus (N/m)

Δp = pressure difference (N/m)

ν = Poisson coefficient



Disc

$$u = \frac{\pi}{4}\rho t \omega^2 R^4$$

$$\sigma_{max} = \frac{1}{8}(3+\nu)\rho\omega^2 R^2$$

Rings

$$u = \pi \rho t \omega^2 R^3 x$$

$$\sigma_{max} = \rho R^2 \omega^2$$

u = energy (J)

ω = angular speed (rad/s)

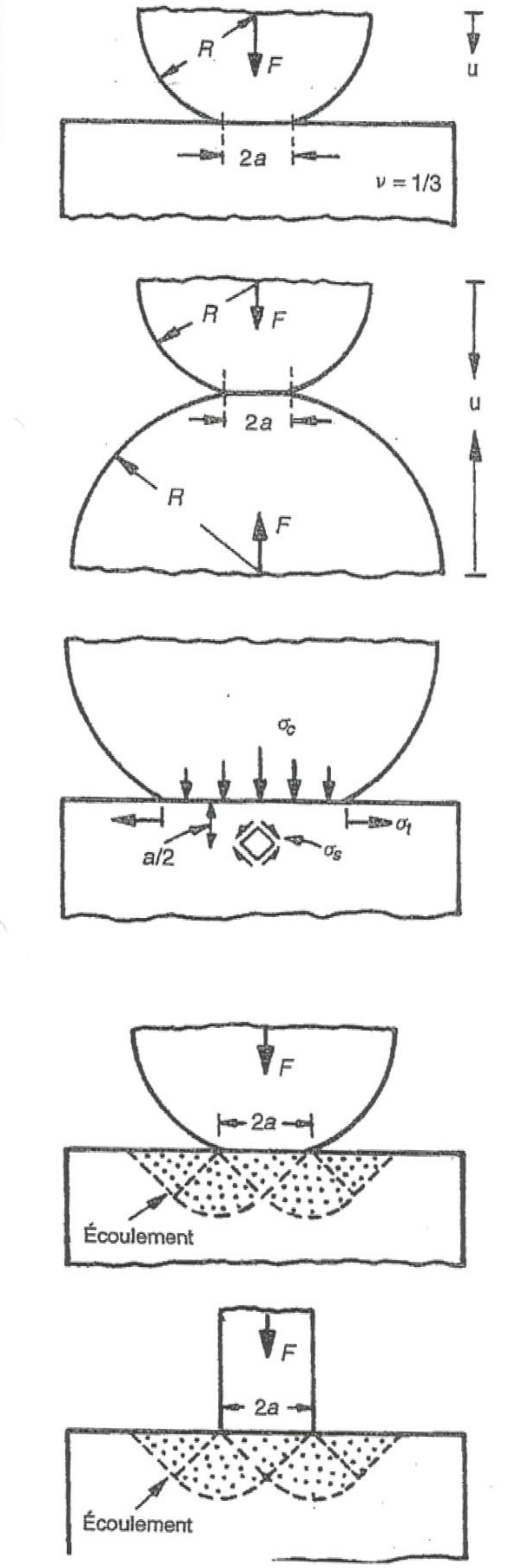
ρ = density (kg/m^3)

When surfaces come into contact, they touch each other in one or more discrete points. If the surfaces are submitted to loads, the contacts flatten elastically and the surfaces of contact increase until failure occurs: by squashing (compressive stress), by traction (traction stress), by plasticity (shear stress). The boxes below give the main results concerning the radius a of the contact zone, the displacement from centre to centre u, and the maximal values of the stresses.

The first box corresponds to the case of a sphere on a plane surface, both having the same Young modulus and a Poisson coefficient of one third.

The second box gives the results in the most general case, for two elastic spheres of radius R_1 and R_2, with, respectively, Young modulus and Poisson coefficient (E_1,ν_1) and (E_2,ν_2) pressed on each other by a force F.

If the shear stress exceeds the shear yield stress $\frac{R_e}{2}$, a plastic zone appears under the centre of the contact at a depth of $\sim\frac{a}{2}$ and spreads to form an entire plastic zone (see the last two schemes). When the material reaches this state, the pressure stress's value is approximately three times equivalent to the yield stress, as indicated in the last box.



$$for\ \nu = \frac{1}{3} \begin{cases} a = 0.7\left(\frac{FR}{E}\right)^{1/3} \\ u = 1.0\left(\frac{F^2}{E^2 R}\right)^{1/3} \end{cases}$$

$$a = \left(\frac{3}{4}\frac{F}{E^*}\frac{R_1 R_2}{(R_1 + R_2)}\right)^{1/3}$$

$$u = \left(\frac{9}{16}\frac{F^2}{{E^*}^2}\frac{(R_1 + R_2)}{R_1 R_2}\right)^{1/3}$$

$$(\sigma_c)_{max} = \frac{3F}{2\pi a^2}$$

$$(\sigma_s)_{max} = \frac{F}{2\pi a^2}$$

$$(\sigma_t)_{max} = \frac{F}{6\pi a^2}$$

$R_1 R_2$ radii of the spheres (m)

$E_1 E_2$ Young modulii of the spheres (N/m^2)

$\nu_1 \nu_2$ Poisson coefficient

F load (N)

a radius of contact (m)

u displacement (m)

σ stress (N/m^2)

σ_y yield stress (N/m^2)

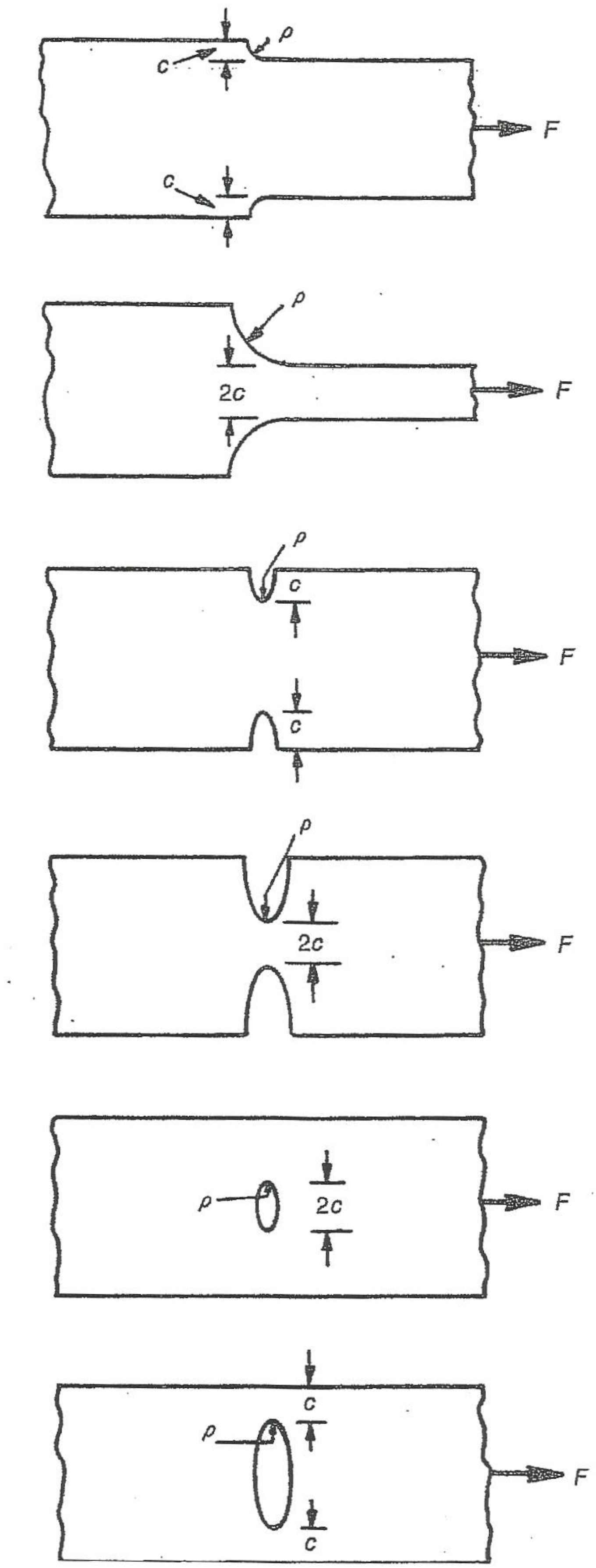
$$E^* = \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)^{-1}$$

$$\frac{F}{\pi a^2} = 3\sigma_y$$

The stresses and strain are concentrated around holes, cracks, or in the neighborhood of where the sections vary a lot. The plastic flow, the failure and the cracking by fatigue start at these points. The local stresses at the point of concentrated stresses can be computed numerically, but it is usually unnecessary. We can simply estimate the value by using the equation indicated here opposite.

The stress concentration caused by a variation of the section disappears at distances that are in the range of the characteristic dimension of cross section variation (defined in a more detailed way here below). This is an illustration of the principle of Saint-Venant. It means that one can find the local maximal stress in a structure by determining the distribution of the nominal stresses, by neglecting local discontinuities (e.g. holes or grooves), then by multiplying nominal stress by a stress concentration factor. An approached value of these factors is given by the equation. In this expression, σ_{nom} is defined as the load divided by the smallest transversal section of the piece, ρ is the smallest radius of curvature of the hole or of the groove and c is the characteristic dimension: the smallest value between the half of the thickness of remaining ligament, half of the length of the inner crack, the length of the crack which attains the surface, or the height of the step, depending on which is the smallest. The schemes show examples of such cases. The factor α is more or less equivalent to one half in traction, but is closer to two in torsion and in bending. But if it is not rigorously exact, this equation provides sufficiently approached values for many design problems.

The maximal stress is limited by plasticity or failure. When plasticity process starts, the strain concentration rapidly increases while the stress concentration stays constant. The strain concentration becomes huge, and can become non negligible at mid distance (the Saint-Venant principle becomes invalid).



$$\frac{\sigma_{max}}{\sigma_{nom}} = 1 + \alpha\left(\frac{c}{\rho}\right)^{1/2}$$

F = force (N)

A_{min} = minimal section (m^2)

$\sigma_{nom} = \frac{F}{A_{min}}$ (N/m^2)

ρ = radius of curvature (m)

c = characteristical dim. (m)

$$\alpha \approx \begin{cases} 0.5\ (tension) \\ 2.0\ (torsion) \end{cases}$$

Acute cracks (that is to say the concentrations of stress with the extremity of its radius of curvature that have the atomic size) concentrate more the stresses than the curved ones. As a first approximation, the local stress diminishes by a $\frac{1}{r^{1/2}}$ law, with r as the radial distance of the extremity of the crack. A traction stress σ, applied perpendicularly to the plane of the crack of length $2a$, and contained in an infinite plane (upper scheme), initiates a local stress field of traction σ_l in the plane of the crack:

$$\sigma_l = \frac{C\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}$$

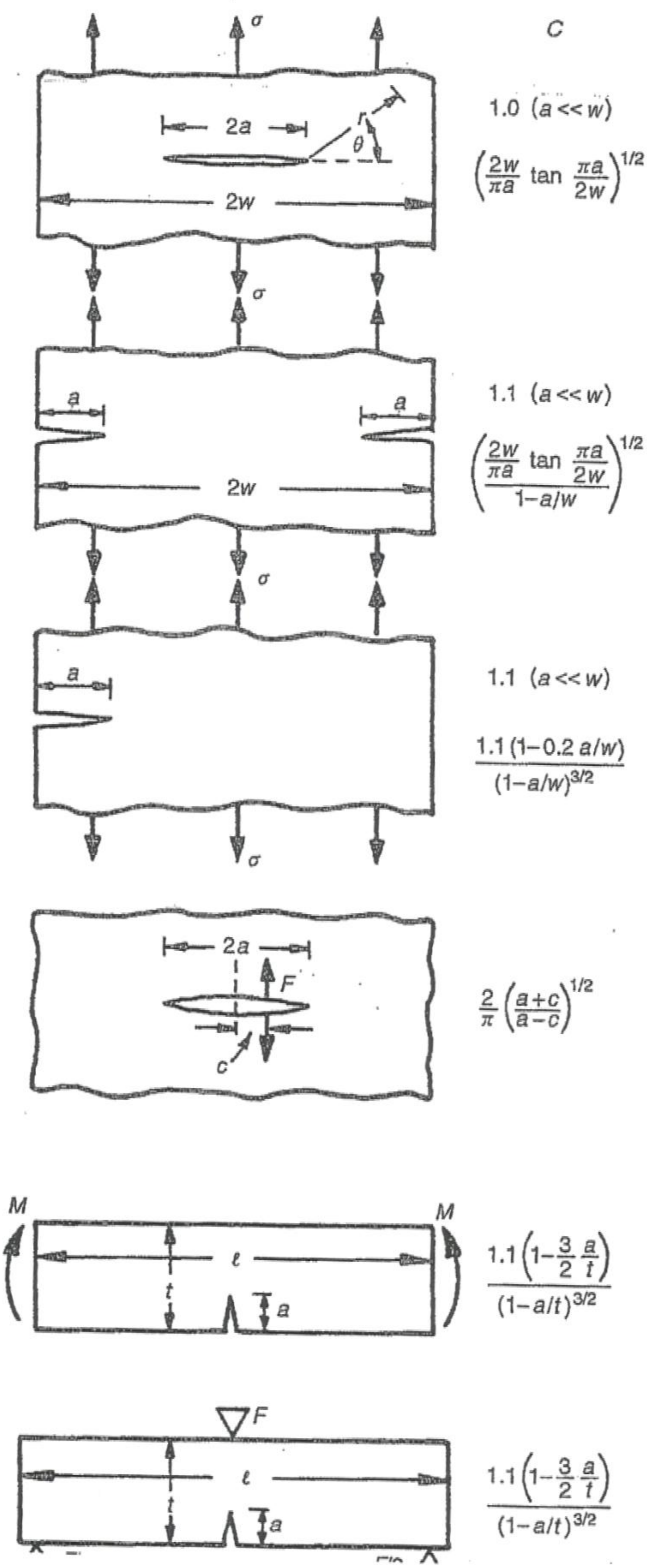
r is measured from the crack tip in the plane $\theta = 0$, and C as a constant.

The stress intensity factor (SIF) in mode I is defined as:

$$K_I = C\sigma\sqrt{\pi a}$$

The values of the constant C for different modes of loading are indicated on the figure. The crack propagates when $K_I > K_{Ic}$ where K_{Ic} is the fracture toughness.

When the length of the crack is small compared to the other dimensions of the sample and compared to the distance on which the stress is applied, C is equal to 1 for an internal crack and is equal to 1.1 for a breaking surface crack. While the crack extends in a uniformly loaded component, it will interact with the free surfaces, giving the correction factors here below. Additionally, if the stress field is not uniform (as in a beam in elastic bending), C is no more equal to 1; two examples are shown here opposite. The C factors indicated here are approximations valid only for small cracks, but they are not valid when the extremities of the crack are near the borders of the sample. These approximations are sufficient for design calculations. More precise ones and particular geometries give rise to C factors that can be found in the references listed in the bibliography.



$$K_1 = C\sigma\sqrt{\pi a}$$

Failure when $K_1 > K_{1C}$

$K_1 =$ stress intensity ($\mathrm{N/m^{3/2}}$)

$\sigma =$ applied stress ($\mathrm{N/m^2}$)

$F =$ load (N)

$M =$ torque (Nm)

$a =$ half length of crack

$=$ length of a breaking surface crack (m)

$w =$ half width (centre) (m)

$=$ breaking surface width (m)

$b =$ sample width (m)

$t =$ sample thickness (m)

Punctual load :

$$\sigma = \frac{F}{2ab}$$

Torque :

$$\sigma = \frac{6M}{bt}$$

3 points bending :

$$\sigma = \frac{3Fl}{2bt}$$

Thin wall pressure tanks are considered as membranes. This assumption stays valid when $t < \frac{b}{4}$. The stresses in the wall are given here opposite. They are almost independent from the radial distance r. The stresses in the tangential plane to the wall, σ_θ and σ_z for a cylinder and σ_θ and σ_φ for a sphere, are equal to the internal pressure amplified by a factor $\frac{b}{t}$ or $\frac{b}{2t}$, depending on the geometry. The radial stress σ_r is equal to the mean between the external and internal stresses, here $\frac{p}{2}$. When an external pressure is added to the problem, the equations stay valid if we replace p by $(p - p_e)$.

In thick wall pressure tanks, the stresses vary with the radial distance r between the external and internal surfaces and are more important on the inner surface. The equations can be adapted to the case of internal and external pressure by noticing that, when these two pressures are equal, the stress field in the wall is:

$$\sigma_\theta = \sigma_r = -p \quad \text{(cylinder)}$$

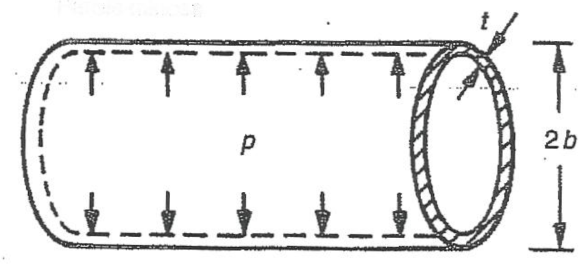
$$\sigma_\theta = \sigma_r = \sigma_\varphi = -p \quad \text{(sphere)}$$

It permits to evaluate the term due to the external pressure. It is not possible here to simply replace p by $(p - p_e)$.

The pressure tanks suffer plasticity when the equivalent Von Mises stress reaches the yield stress R_e. These tanks fail if the biggest traction stress reaches the rupture stress σ_f, with :

$$\sigma_f = \frac{C K_{IC}}{\sqrt{\pi a}}$$

with K_{IC} as the fracture toughness of the material, a as the half length of the crack and C as a constant (see A.10).

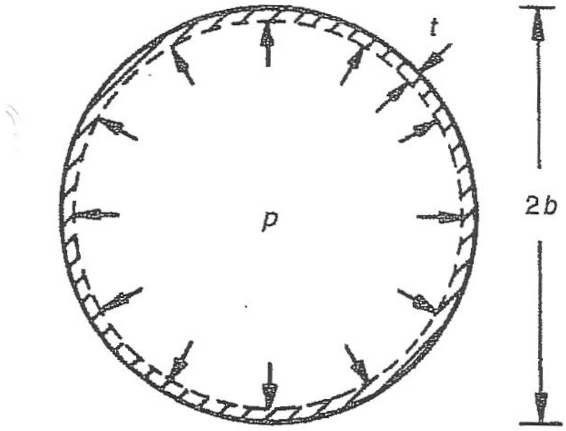


Cylinder

$$\sigma_\theta = \frac{pb}{t}$$

$$\sigma_r = -p/2$$

$$\sigma_z = \frac{pb}{2t}$$



Sphere

$$\sigma_\theta = \sigma_\varphi = \frac{pb}{2t}$$

$$\sigma_r = -p/2$$

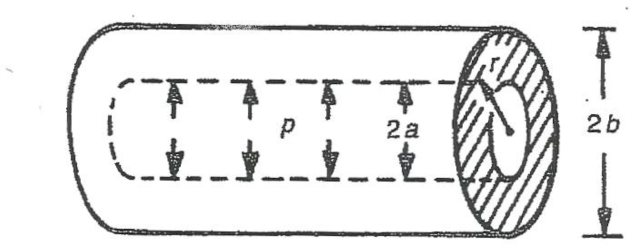
p = pression ($\mathrm{N/m^2}$)

t = thickness of the wall (m)

a = internal radius (m)

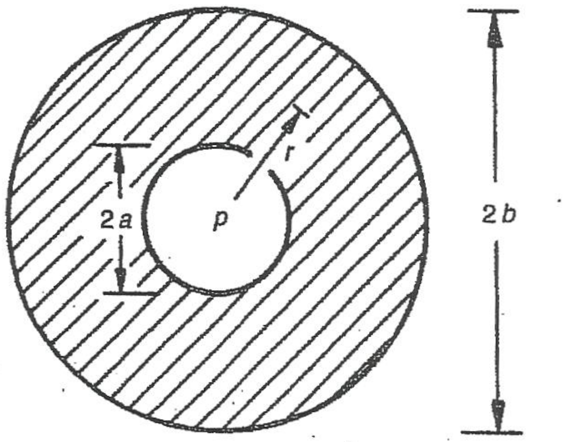
b = external radius (m)

r = radial coordinate (m)



$$\sigma_\theta = \frac{pa^2}{r^2}\left(\frac{b^2 - r^2}{b^2 - a^2}\right)$$

$$\sigma_r = \frac{pa^2}{r^2}\left(\frac{b^2 + r^2}{b^2 - a^2}\right)$$



$$\sigma_\theta = \sigma_\varphi = \frac{pa^3}{2r^3}\left(\frac{b^3 + 2r^3}{b^3 - a^3}\right)$$

$$\sigma_r = \frac{pa^3}{r^3}\left(\frac{b^3 - r^3}{b^3 - a^3}\right)$$

Any vibrating non damped system that vibrates at one of its Eigen frequency can be modeled by a simple masse m linked to a spring of stiffness k. The lowest Eigen frequency of such a system is:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

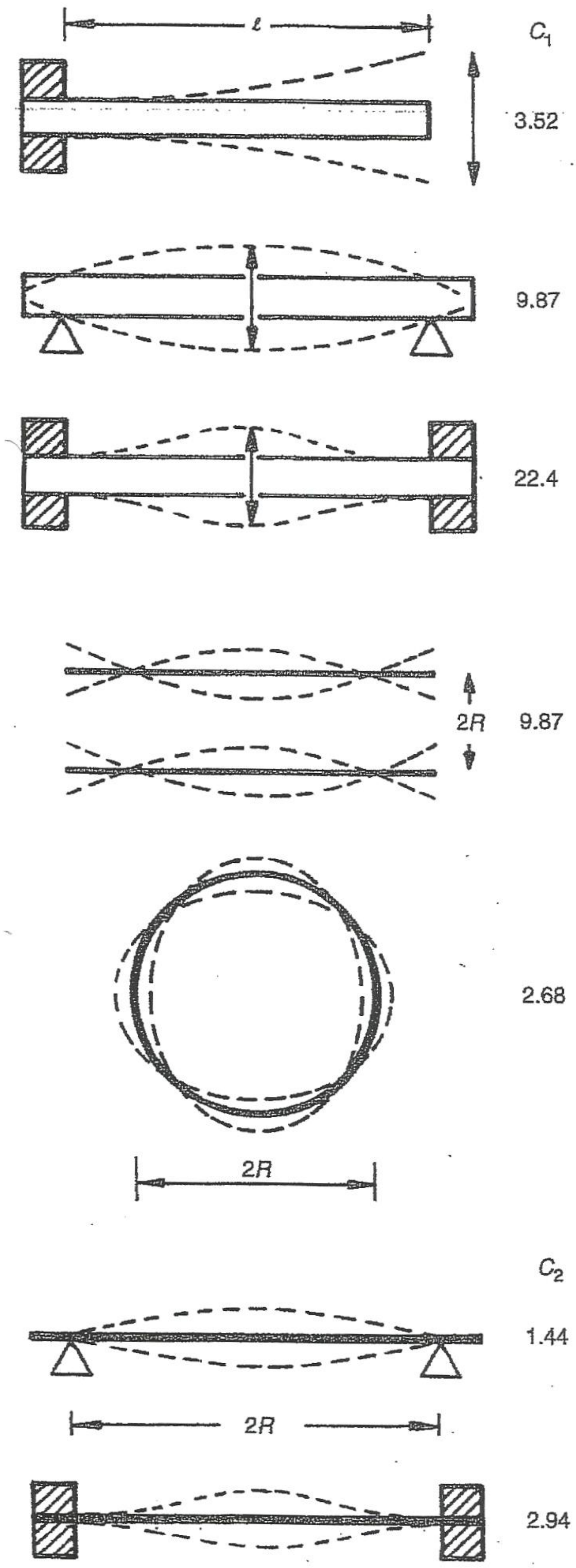
Each case has its own m and k value that can be estimated with a sufficient precision to establish an approximate model. The higher Eigen frequencies are simply harmonics of the one here above.

The first box here below gives the smallest Eigen frequencies of bending beams for different boundary conditions. For instance, the first one can be estimated by assuming that the effective mass of the beam is equal to the quarter of its real mass:

$$m = \frac{m_0 l}{4}$$

with m_0 as the linear mass of the beam. This leads to a 0.2 % estimation of the exact value. The vibrations of a tube have a similar form. The ones at the circumference can be expressed in a rough way by “developing” the tube, with the aim of treating it as a plate in vibration with simple supports at its four extremities.

The second box gives the Eigen frequencies of circular plane discs, simply lying on their rim of clamped. The discs that have their two faces curved are more rigid and thus, their Eigen frequencies are higher.



Beams, tubes:

$$F_1 = \frac{C_1}{2\pi}\sqrt{\frac{EI}{m_0 l^4}}$$

f = Eigen frequency (Hz)

$m_0 = \rho A$ = linear mass (kg/m)

ρ = density (kg/m^3)

A = area of the section (m^2)

I = see table A1

$$\begin{cases} with\ A = 2\pi R \\ I = \pi R^3 t \end{cases}$$

$$\left\{ with\ A = \frac{lt^3}{12} \right.$$

Discs:

$$f_1 = \frac{C_2}{2\pi}\sqrt{\frac{Et^3}{m_1 R^4 (1-\nu^2)}}$$

$m_1 = \rho t$ = surface mass (kg/m^2)

t = thickness (m)

R = radius (m)

ν = Poisson coefficient

For temperatures higher than a third of the fusion point of the material $T_f/3$, loaded materials are subjected to creep. It is convenient to characterize the creep of a material by its behavior under a traction stress σ at a temperature T. In these conditions, the stationary strain rate often varies as a power law of the stress and as an exponential law of the temperature:

$$\dot{\varepsilon} = A\left(\frac{\sigma}{\sigma_0}\right)^n e^{-\frac{Q}{RT}}$$

with Q as an activation energy, A as a kinetic constant, and R as the ideal gas constant. At a constant temperature, the law becomes:

$$\dot{\varepsilon} = \dot{\varepsilon}_0\left(\frac{\sigma}{\sigma_0}\right)^n$$

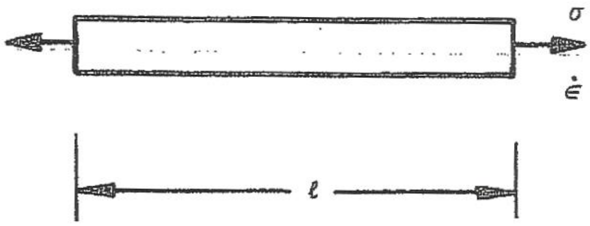
With $\dot{\varepsilon}_0$, σ_0, and n as creep constants.

The creep behavior is described here opposite; the strain rate of a beam, the displacement speed of a punch, and the density variation of spherical and cylindrical tanks under pressure are to be found there.

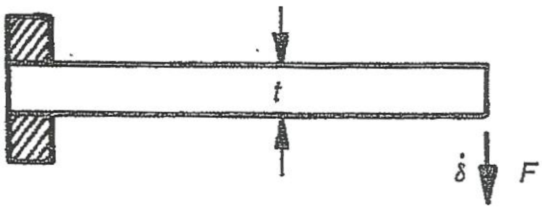
An extended creep induces damage accumulation that leads to failure, after a duration t_f, following the approximate law:

$$t_f\dot{\varepsilon} = C$$

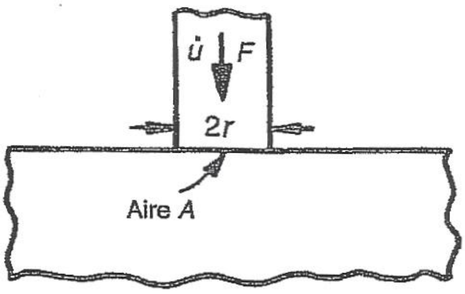
with C as a characteristic constant of the material. Ductile materials in creep have C values comprised between 0.1 and 0.5; C values for fragile materials can reach 0.01 .



$$\dot{\varepsilon} = \dot{\varepsilon}_0 \left(\frac{\sigma}{\sigma_0}\right)^n$$

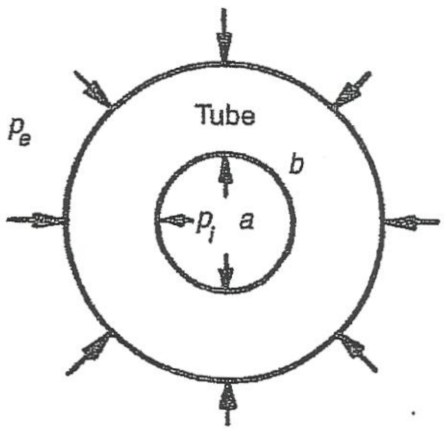


$$\dot{\delta} = \frac{2\dot{\varepsilon}_0}{n+2}\left(\frac{(2n+1)}{n\sigma_0}\,\frac{E}{bt}\right)^n \left(\frac{l}{t}\right)^{n+1} l$$

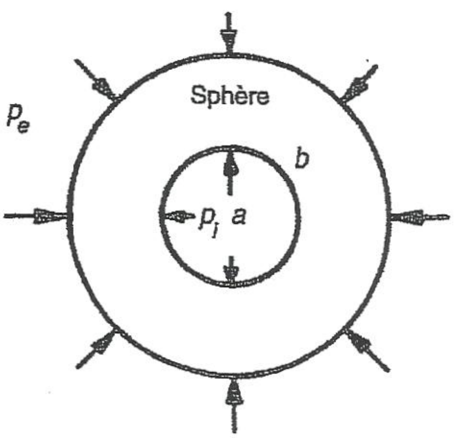


$$\dot{u} = \mathrm{C}_1 \dot{\varepsilon}_0 \sqrt{\mathrm{A}} \left(\frac{C_2 F}{\sigma_0 A}\right)^n$$

$$\dot{\rho} = 2\dot{\varepsilon}_0 \frac{\rho(1-\rho)}{(1-(1-\rho)^{1/n})^n} \left(\frac{2}{\mathrm{n}} \frac{(\rho_e - \rho_i)}{\sigma_0}\right)^n$$



$$\dot{\rho} = \frac{3}{2}\dot{\varepsilon}_0 \frac{\rho(1-\rho)}{(1-(1-\rho)^{1/n})^n} \left(\frac{3}{2\mathrm{n}} \frac{(\rho_e - \rho_i)}{\sigma_0}\right)^n$$



σ = stress (N/m^2)

F = force (N)

$\dot{\delta}, \dot{u}$ = displacement speed (m/s)

$n, \dot{\varepsilon}_0, \sigma_0$ = creep constants

l, b, t = dimensions (m)

a, b = radius (m)

ρ = relative density, $\frac{b^3 - a^3}{b^3}$

C_1, C_2 = constants

Conduction, convection or radiation can limit heat fluxes. The constitutive equations for each of these modes are listed here opposite.

The first equation corresponds to the First Fourier Law which describes the conductive heat flux in steady state. This law uses the thermal conductivity λ.

The second equation corresponds to the Second Fourier Law which governs the unsteady conductive problems. This law uses the thermal diffusivity a defined as:

$$a = \frac{\lambda}{\rho C}$$

with ρ as the density, and C as the specific heat at constant pressure.

The solutions of these two differential equations are given in section A.15.

The third equation describes the convective heat transfer which limits the heat flux when the Biot Number is smaller than 1:

$$B_i = \frac{hs}{\lambda} < 1$$

with h as the transfer heat coefficient and s as a characteristic dimension of the sample. On the contrary, when $B_i > 1$, the heat flux is limited by the conduction equation.

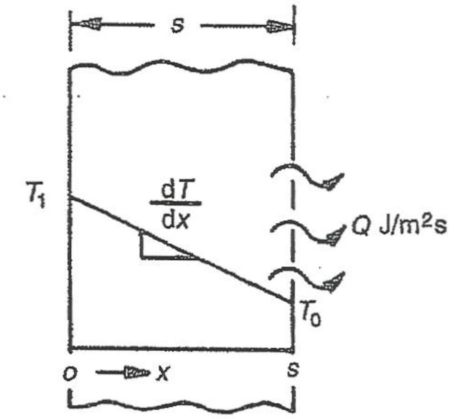
The last equation traduces the Stefan-Boltzmann Law for radiation heat transfers. The emissivity ε equals 1 for black bodies and is lower for other bodies.

The matter diffusion is governed by a pair of differential equations which are similar to Fourier Laws and have similar solutions. These equations are written under the following usual form:

$$J = -D\nabla C = -D\frac{dC}{dx} \text{ (steady state)}$$

$$\frac{\partial C}{\partial t} = D\nabla C^2 = D\frac{\partial^2 C}{\partial x^2} \text{ (unsteady state)}$$

with J as the flux, C as the concentration, x as the distance and t as the time. The solution of these equations is given in the following section.



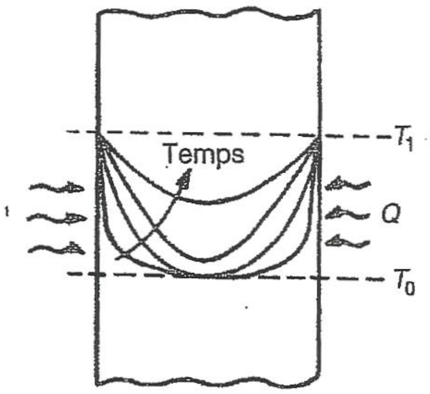
$$Q = -\lambda \nabla T = -\lambda \frac{dT}{dx}$$

Q = heat flux (J/m^2s)

T = temperature (K)

x =distance (m)

λ = thermal conductivity (W/mK)



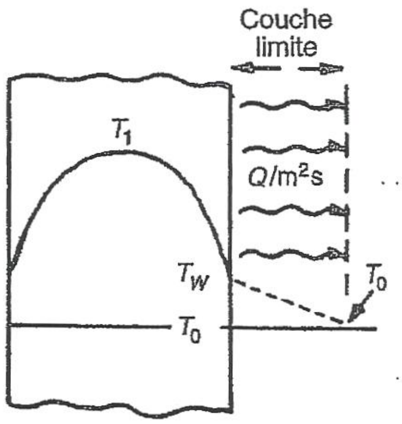
$$\frac{\partial T}{\partial t} = a\nabla^2 T = a\frac{\partial^2 T}{\partial x^2}$$

t = time (s)

ρ = density (kg/m^3)

C = specific heat (J/m^3K)

a = thermal diffusivity (m^2/s)



$$Q = h(T_w - T_0)$$

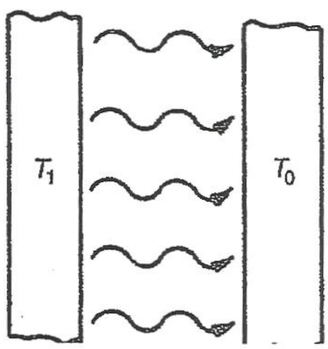
T_w = surface temperature (K)

T_0 = fluide temperature (K)

h = transfer coefficient (W/m^2K)

= $[5 - 50]$ in air

= $[1000 - 5000]$ in water



$$Q = \varepsilon\sigma(T_1^4 - T_0^4)$$

ε = emissivity (= 1 for black bodies)

σ = Stefan constant

= $5.67 \cdot 10^{-8}$ (W/m^2K^4)

One can find the solution of the diffusion equation for a certain number of usual geometries. It is quite useful to know them because a lot of real problems can be approximately solved by using one of them.

In steady state, the temperature/concentration profile does not vary with respect to time. This can be seen in the first box here opposite. The solutions are given below the equations for uniaxial (radial for cylinders/spheres) flow. The solution for a particular problem is found by applying the boundary conditions, in order to determine the constants A and B.

One can find the solution for matter fluxes by replacing T by the concentration C and λ by the coefficient of diffusion D.

The second box gives the equations for unsteady state problems, making the assumption that the diffusivity (a or D) does not depend on the position. The solutions for the temperature profiles $T(x,t)$ or for the concentration profiles $C(x,t)$ are given here below.

The first equation applies for a thin film body: a thin slice at the temperature T_1 (or at the concentration C_1) is in-between the two semi-infinite blocs at the temperature T_0 (or C_0), at $t = 0$ (s); then the flow begins.

The second equation is valid for two semi-infinite blocs that initially had a temperature T_1 and T_0 (C_1 and C_0), assembled at $t = 0$ (s).

The last equation applies for a sinusoidal profile of temperature/concentration of amplitude A at $t = 0$ (s).

Note that every unsteady state problem has:

- a characteristic time constant t^* equal to:

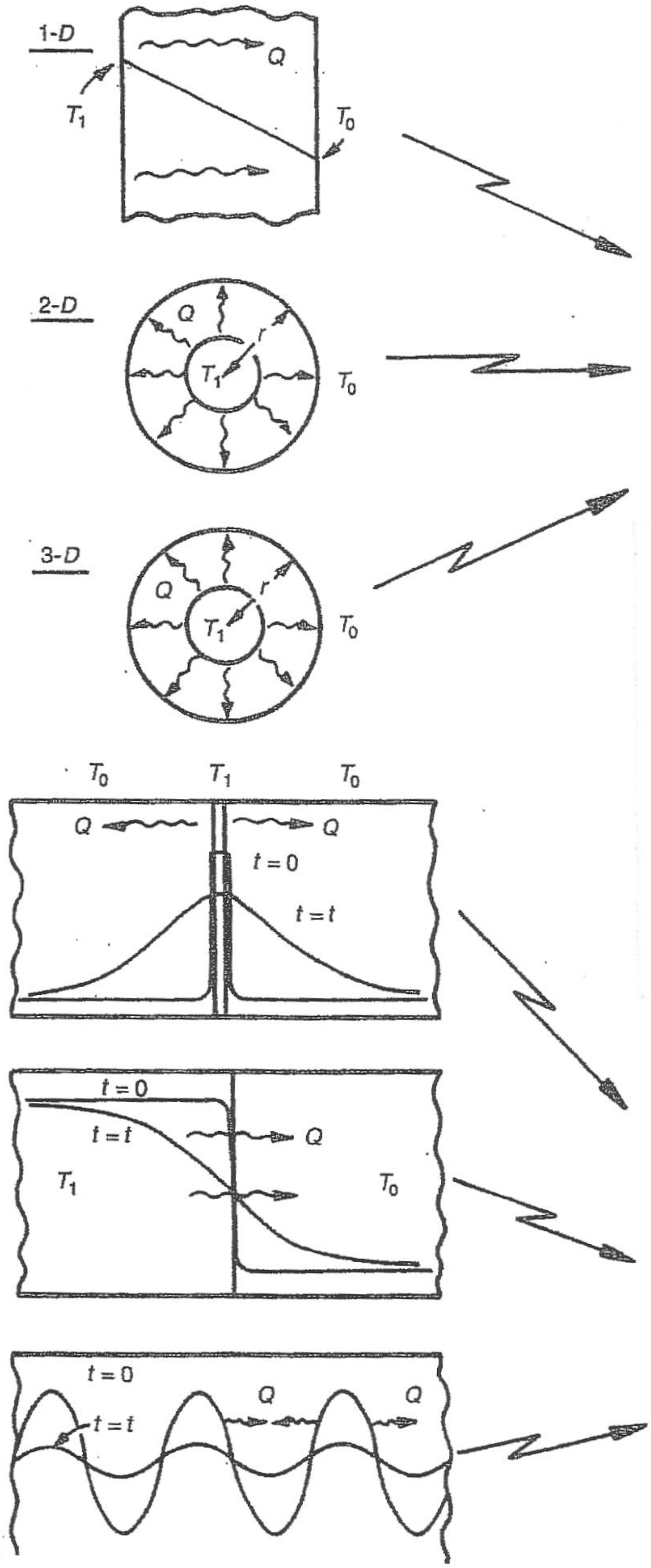
$$t^* = \frac{x^2}{\beta a} \text{ or } \frac{x^2}{\beta D}$$

with x as a dimension of the sample.

- a characteristic dimension x^* equal to:

$$x^* = \sqrt{\beta a t} \text{ or } \sqrt{\beta D t}$$

with t as the time scale of the observation, and $1 < \beta < 4$ depending on the geometry.



Steady State

$$\lambda \nabla^2 T = 0$$

$$D \nabla^2 C = 0$$

$\lambda T(x) = \; Ax + B(1-D)$

$\lambda T(r) = \; Alnt + B(2-D)$

$\lambda T(r) = \frac{A}{r} + B(3-D)$ etc

$A, B =$ integration constants

$x =$ linear distance (m)

$r =$ radial distance (m)

Unsteady State

$$\frac{\partial \mathrm{T}}{\partial \mathrm{t}} = a \nabla^2 T$$

$$\frac{\partial \mathrm{C}}{\partial \mathrm{t}} = D \nabla^2 C$$

$T(x,t) = \frac{A}{\sqrt{at}} \exp\left\{-\left(\frac{x^2}{4at}\right)\right\} + B$

$T(x,t) = A\left(1 + erf\left\{\frac{x}{2\sqrt{at}}\right\}\right) + B$

$T(x,t) = Asin(\lambda x), \exp\{-\lambda^2 at\}$, etc

B • Performance indexes

B.1 Introduction

The performance p of a component is measured by a performance equation which contains groups of variables in connection with the properties of the material. These groups are the performances indexes. Sometimes the "group" is reduced to a single property. For instance, the performance of a beam is measured by its rigidity and its performance equation only contains one property quantity, the Young's modulus E. Generally, the performance equation contains a unique group of two variables or more. Specific rigidity $\frac{E}{\rho}$ and specific strength $\frac{\sigma_f}{\rho}$ are often encountered.

These indexes are the key to the optimization process in order to select the right material for the dedicated application. The details about the selection method using the indexes are given in chapter 6. Softwares such as CES exist and implement this method. In this appendix, you will find several indexes for usual applications.

B.2 Use of the indexes

B.2.1 Materials choice

Components fulfill different functions: to resist safety loads, to transmit heat, to store energy, to insulate, and many more. Each of these functions has its own performance index. The materials which present high values for an appropriated index maximize the goal performance of the component. For the reasons detailed in chapter 6, the index of a material is usually independent from the details of its design (in terms of geometries, loads, …).

B.2.2 New materials or substitutes

A new material will have potential applications in the functions in which it presents exceptionally high values for a required index. The calculation of these indexes and their comparison with existing materials ones permits to identify right applications for new materials. The same reasoning can be applied to identify right substitutes for currently used materials.

Table B.1 – Design at minimal weight (or cost, energy content, environmental impact), limited by the material rigidity

Functions and constraints[1]	**Maximize**[2]
Bar (in tension)	
rigidity and length specified, free cross section	E/ρ
Shaft (in torsion)	
rigidity, length, and geometry specified, free cross section	$G^{1/2}/\rho$
rigidity, length, and external radius specified, free wall thickness	G/ρ
rigidity, length, and wall thickness specified, free external radius	$G^{1/3}/\rho$
Beam (in bending)	
rigidity, length, and geometry specified, free cross section	$E^{1/2}/\rho$
rigidity, length, and height specified, free width	E/ρ
rigidity, length, and width specified, free height	$E^{1/3}/\rho$
Column (in compression)	
buckling load, length, and geometry specified, free cross section	$E^{1/2}/\rho$
Plate (in flexion)	
rigidity, length, and width specified, free thickness	$E^{1/3}/\rho$
Plate (in compression, buckling)	
crushing load, length, and width specified, free thickness	$E^{1/3}/\rho$
Pressure tank	
elastic deformation, length, and width specified, free wall thickness	E/ρ
Spherical shell (internal pressure)	
elastic deformation, and radius specified, free wall thickness	$E/(1-\nu)\rho$

[1] To minimize the cost instead of the weight, one can still use the index for the minimal weight but it is necessary to replace density by $C_m.\rho$ (where C_m is the cost by kilo). To minimize the energy content, one can replace density by $q.\rho$ (where q is the energy content by kilo). To minimize the environmental impact, one can replace density by $l_e.\rho$ (where l_e is the eco-index of the material).

[2] E = Young's modulus for traction, bending, and buckling. G = shear modulus. ρ = density.

Table B.2 – Design at minimal weight (or cost, energy content, environmental impact), limited by the material strength

Functions and constraints[1]	Maximize[2]
Bar (in tension)	
rigidity and length specified, free cross section	σ_f/ρ
Shaft (in torsion)	
load, length, and geometry specified, free cross section	$\sigma_f^{2/3}/\rho$
load, length, and external radius specified, free wall thickness	σ_f/ρ
load, length, and wall thickness specified, free external radius	$\sigma_f^{1/2}/\rho$
Beam (in bending)	
load, length, and geometry specified, free cross section	$\sigma_f^{2/3}/\rho$
load, length, and height specified, free width	σ_f/ρ
load, length, and width specified, free height	$\sigma_f^{1/2}/\rho$
Column (in compression)	
buckling load, length, and geometry specified, free cross section	σ_f/ρ
Plate (in flexion)	
rigidity, length, and width specified, free thickness	$\sigma_f^{1/2}/\rho$
Plate (in compression, buckling)	
crushing load, length, and width specified, free thickness	$\sigma_f^{1/2}/\rho$
Pressure tank	
elastic deformation, length, and width specified, free wall thickness	σ_f/ρ
Spherical shell (internal pressure)	
elastic deformation, and radius specified, free wall thickness	σ_f/ρ
Inertia wheel, rotating disc	
max stored volumetric energy, and speed specified	ρ
max stored mass energy, no failure	σ_f/ρ

[1] σ_f = mechanical strength (yield stress for metals and ductile polymers; traction strength for ceramics, glasses and fragile polymers loaded in traction; failure modulus for materials loaded in bending). ρ = density.

[2] For fatigue design, replace σ_f by the endurance limit σ_e.

Table B.3 – Design limited by the strength: springs, hinges, etc

Functions and constraints	**Maximize**
Spring	
max volumetric stored elastic energy, no failure	σ_f^2 / E
max masse stored elastic energy, no failure	$\sigma_f^2 / E\rho$
Elastic hinge	
min curvature radius (max flexibility without failure)	σ_f / E
Knife, pivot	
min contact area, max contact load	σ_f^3 / E^2 and H
Seal	
max adaptability, limited contact pressure	$\sigma_f^{3/2} / E$ and $1/E$
Diaphragm	
max displacement under a specified pressure/force	$\sigma_f^{3/2} / E$
Rotating drum, centrifuge	
max angular speed, fixed radius, free wall thickness	σ_f / ρ

Table B.4 – Design limited by vibrations

Functions and constraints	Maximize[(1)]
Bar (traction), column (compression)	
max longitudinal vibration frequencies	E/ρ
Beam (all dimensions specified)	
max bending vibration frequencies	E/ρ
Beam (length and rigidity specified)	
max bending vibration frequencies	$E^{1/2}/\rho$
Plate (all dimensions specified)	
max bending vibration frequencies	E/ρ
Plate (length, width, and rigidity specified)	
max bending vibration frequencies	$E^{1/3}/\rho$
Bar, column, beam, plate (rigidity specified)	
bar: min external longitudinal excitation	$\eta E/\rho$
beam: min external bending excitation	$\eta E^{1/2}/\rho$
plate: min external bending excitation	$\eta E^{1/3}/\rho$

[(1)] η is the damping capacity.

Table B.5 – Damage tolerant design

Functions and constraints	Maximize[(1)]
Bar (traction), shaft (torsion), beam (bending)	
max strength and damage tolerance, load controlled design	K_{Ic} and σ_f
max strength and damage tolerance, displacement controlled design	K_{Ic}/E and σ_f
max strength and damage tolerance, energy controlled design	${K_{Ic}}^2/E$ and σ_f
Internally pressured container	
plasticity occurs before failure (ductile failure)	K_{Ic}/σ_f
leaking occurs before failure	${K_{Ic}}^2/\sigma_f$

[(1)] K_{Ic} is the resistance to crack propagation.

Table B.6 – Thermal and thermo-mechanical design

Functions and constraints	Maximize[1]
Insulating material	
min thermal flux in steady state, specified thickness	$1/\lambda$
min increase of temperature during a given duration, specified thickness	$1/a = \rho C_p/\lambda$
min total consumed energy by a thermal cycle (oven, etc)	$\sqrt{a}/\lambda = \sqrt{\lambda C_p \rho}$
Heat storage material	
max stored energy by unit cost of material	C_p/C_m
max stored energy during an increase of temperature during a given duration	$\lambda/\sqrt{a} = \sqrt{\lambda C_p \rho}$
measure device	
min thermal displacement under a given thermal flux	λ/α
Resistance to thermal choc	
max surface temperature variation, no failure	$\sigma_f/E\alpha$
Heat exchanger	
max volumetric heat flux, limited expansion	$\lambda/\Delta\alpha$
max masse heat flux, limited expansion	$\lambda/\rho\Delta\alpha$
Heat exchanger (limited by pressure)	
max heat flux by unit area, no failure under Δp	$\lambda\sigma_f$
max masse heat flux, no failure under Δp	$\lambda\sigma_f/\rho$

[1] λ is the thermal conductivity, a the thermal diffusivity, C_p is the specific heat capacity, C_m is the cost/kg of material, α is the coefficient of thermal expansion.

Table B.7 – Electro-mechanical design

Functions and constraints	Maximize
Supply bus	
min cost along the lifetime, driver for high currents	$1/\rho\rho_e C_m$
Electromagnet winding	
max short impulse, no mechanical failure	R_e
max field and impulse duration, limited increase of temperature	$C_p\rho/\rho_e$
high speed electric engine winding	
max rotational speed, no fatigue failure	R_e/ρ_e
min ohmic losses, no fatigue failure	$1/\rho_e$
Relay arm	
min time before response, no fatigue failure	$R_e/E\rho_e$
min ohmic losses, no fatigue failure	$R_e^2/E\rho_e$